

Highlights of Noncommutative Spectral Geometry

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Abstract. A summary of noncommutative spectral geometry as an approach to unification is presented. The rôle of the doubling of the algebra, the seeds of quantization and some cosmological implications are briefly discussed.

1. Introduction

In the various attempts to quantize gravity, it is either assumed a purely gravitational theory without any matter fields, or it is considered that the interaction between gravity and matter is the most important element of the dynamics. I will adopt here the second approach.

Much below the Planck energy scale, gravity can be considered as a classical theory, and the laws of physics can be described to a good approximation by an effective action and continuum fields. As the energies however approach the Planck scale, the quantum nature of space-time becomes apparent, and the simple prescription, dictating that physics can be described by the sum of the Einstein-Hilbert and the Standard Model (SM) action ceases to be valid. In the framework of NonCommutative Spectral Geometry (NCSG) [1], gravity and the SM fields were put together into matter and geometry on a noncommutative space made from the product of a four-dimensional standard commutative manifold by a noncommutative internal space.

In what follows, I will briefly present the elements of NCSG as an approach to unification [2] and highlight the relation between the doubling of the algebra and the gauge fields [3], an essential element to make the link with the SM. I will then discuss how the doubling of the algebra is related to dissipation, which incorporates the seeds of quantization [3]. I will finally discuss briefly some cosmological consequences [4, 5, 6, 7, 8], since the model lives by construction in high energy scales, offering a natural framework to address early universe cosmology.

2. Elements of noncommutative spectral geometry

NCSG [9, 10] is based on a two-sheeted space, made from the product of a four-dimensional smooth compact Riemannian manifold \mathcal{M} (a continuous geometry for space-time), by a discrete noncommutative space \mathcal{F} (an internal geometry for the SM) composed by only two points. The noncommutative nature of the discrete space \mathcal{F} is given by a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, where \mathcal{A} is an involution of operators on the finite-dimensional Hilbert space \mathcal{H} of Euclidean fermions, and \mathcal{D} is a self-adjoint unbounded operator in \mathcal{H} . All information about space is encoded in the algebra of coordinates \mathcal{A} , which is related to the gauge group of local gauge transformations.

Assuming the algebra \mathcal{A} to be symplectic-unitary, it is [11] $\mathcal{A} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C})$, with $k = 2a$ and \mathbb{H} denoting the algebra of quaternions. The choice $k = 4$ is the first value that produces

the correct number ($k^2 = 16$) of fermions in each of the three generations, with the number of generations being a physical input. While the choice of algebra \mathcal{A} constitutes the main input of the theory, the choice of Hilbert space \mathcal{H} is irrelevant. The operator \mathcal{D} corresponds to the inverse of the Euclidean propagator of fermions, and is given by the Yukawa coupling matrix which encodes the masses of the elementary fermions and the Kobayashi–Maskawa mixing parameters. The SM fermions provide the Hilbert space \mathcal{H} of a spectral triple for the algebra \mathcal{A} , while the bosons are obtained through inner fluctuations of the Dirac operator of the product geometry.

One applies the spectral action principle [12], stating that the bare bosonic Euclidean action is the trace of the heat kernel associated with the square of the Dirac operator and is of the form $\text{Tr}(f(\mathcal{D}/\Lambda))$; f is a cut-off function and Λ fixes the energy scale. This action can be seen *à la* Wilson as the bare action at scale Λ . The fermionic term can be included by adding $(1/2)\langle J\psi, D\psi \rangle$, where J is the real structure on the spectral triple and ψ is a spinor in the Hilbert space of the quarks and leptons. For the four-dimensional Riemannian geometry, the trace is expressed perturbatively in terms of the geometrical Seeley–deWitt coefficients a_n [13]:

$$\text{Tr}(f(D/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4 + \cdots + \Lambda^{-2k} f_{-2k} a_{4+2k} + \cdots . \quad (1)$$

Since its Taylor expansion at zero vanishes, it reduces to

$$\text{Tr}(f(D/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4 ; \quad (2)$$

f plays a rôle only through its momenta f_0, f_2, f_4 , which are three real parameters, related to the coupling constants at unification, the gravitational constant, and the cosmological constant, respectively. The computation of this asymptotic expression results to the full Lagrangian for the SM minimally coupled to gravity, with neutrino mixing and Majorana mass terms.

This purely geometric approach to the SM leads to the correct representations of the fermions with respect to the gauge group of the SM, the Higgs doublet appears as part of the inner fluctuations of the metric, and Spontaneous Symmetry Breaking mechanism arises naturally with the negative mass term without any tuning [1]. The see-saw mechanism is obtained, the 16 fundamental fermions are recovered, and a top quark mass of ~ 179 GeV is predicted [1]. The model also predicts the correct order of magnitude for the Higgs mass. Strictly speaking, the predicted Higgs mass of approximately 170 GeV is ruled out from the experimental data, nevertheless it is rather remarkable that the order of magnitude is correct, given that the NCGS approach based on the particular choice for the algebra \mathcal{A} must be seen as an effective theory.

3. Dissipation and the origin of quantization

The central ingredient in the NCSG, namely the doubling of the algebra $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ acting on the space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is related to dissipation and to the gauge field structure [3].

To highlight the justification of this claim let us consider the equation of the classical one-dimensional damped harmonic oscillator $m\ddot{x} + \gamma\dot{x} + kx = 0$, with time independent m , γ and k , which is a simple prototype of open systems. In the canonical formalism for open systems, the doubling of the degrees of freedom is required in such a way as to complement the given open system with its time-reversed image, thus obtaining a globally closed system for which the Lagrangian formalism is well defined. Considering the oscillator in the doubled y coordinate $m\ddot{y} - \gamma\dot{y} + ky = 0$ and then using the coordinates $x_1(t) = (x(t) + y(t))/\sqrt{2}$ and $x_2(t) = (x(t) - y(t))/\sqrt{2}$, the Lagrangian of this closed system takes the form

$$L = \frac{1}{2m}(m\dot{x}_1 + \frac{e_1}{c}A_1)^2 - \frac{1}{2m}(m\dot{x}_2 + \frac{e_2}{c}A_2)^2 - \frac{e^2}{2mc^2}(A_1^2 + A_2^2) - e\Phi , \quad (3)$$

where we have introduced the vector potential $A_i = (B/2)\epsilon_{ij}x_j$ for $i, j = 1, 2$ with $B \equiv \gamma c/e$ and $\epsilon_{ii} = 0$, $\epsilon_{12} = -\epsilon_{21} = 1$. It describes two particles with opposite charges $e_1 = -e_2 = e$

in the (oscillator) potential $\Phi \equiv (k/2e)(x_1^2 - x_2^2) \equiv \Phi_1 - \Phi_2$ with $\Phi_i \equiv (k/2e)x_i^2$ and in the constant magnetic field \mathbf{B} defined as $\mathbf{B} = \nabla \times \mathbf{A}$.

The doubled coordinate, e.g., x_2 acts as the gauge field component A_1 to which the x_1 coordinate is coupled, and *vice versa*. The energy dissipated by one of the two systems is gained by the other. The gauge field acts as the bath or reservoir in which the system is embedded [3].

The NCSG classical construction carries implicit in the doubling of the algebra the seeds of quantization [3]. 't Hooft has conjectured that, provided some specific energy conditions are met and some constraints are imposed, loss of information might lead to a quantum evolution [14]. By considering the classical damped harmonic oscillator and its time-reversed image, we have shown [3] that the obtained Hamiltonian belongs to the class of Hamiltonians considered by 't Hooft. We have shown [3] that the dissipation term in the Hamiltonian is responsible for the zero point contribution to the energy, which is the signature of quantization.

4. Cosmological consequences

In the low-energy limit the corrections to the background Einstein's equations do not occur at the level of a Friedmann-Lemaître-Robertson-Walker (FLRW) background [4]. One may have naively claimed that this was expected, arguing that in a spatially homogeneous space-time the spatial points are equivalent and any noncommutative effects are then expected to vanish. However, this argument does not apply here; the noncommutativity is incorporated in the internal manifold \mathcal{F} and the space-time is a commutative four-dimensional manifold. The coupling between the Higgs field and the background geometry can no longer be neglected once the energies reach the Higgs scale, in which case the nonminimal coupling of Higgs field to curvature leads to corrections to Einstein's equations even for homogeneous and isotropic cosmological models [4]. The effect of the nonminimal coupling of the Higgs field can be seen in two ways: it leads to an effective gravitational constant, or it increases the Higgs mass [4].

The nonminimal coupling between the Higgs field and the Ricci curvature may turn out to be crucial in early universe cosmology [5, 6]. Such a coupling has been introduced *ad hoc* in the literature, in an attempt to drive inflation through the Higgs field. However, the value of the coupling constant between the scalar field and the background geometry should be dictated by the underlying theory. Actually, even if classically the coupling between the Higgs field and the Ricci curvature could be set equal to zero, a nonminimal coupling will be induced once quantum corrections in the classical field theory are considered.

We have thoroughly investigated [6] where the Higgs field could play the rôle of the inflaton leading to a sufficient period of inflation with induced temperature anisotropies which are in agreement with the current measurements. The Higgs potential is $V(H) = \lambda_0 H^4 - \mu_0^2 H^2$, with μ_0 and λ_0 subject to radiative corrections as functions of energy. For large enough values of the Higgs field, the normalized value of μ_0 and λ_0 must be calculated. We have shown [6] that for each value of the top quark mass there is a value of the Higgs mass where the effective potential is about to develop a metastable minimum at large values of the Higgs field and the Higgs potential is locally flattened. Calculating [6] the renormalization of the Higgs self-coupling up to two-loops, we have constructed an effective potential which fits the renormalization group improved potential around the flat region. There is a very good analytic fit to the Higgs potential around the minimum of the potential, namely [6] :

$$V^{\text{eff}} = \lambda_0^{\text{eff}}(H)H^4 = [a \ln^2(b\kappa H) + c]H^4, \quad (4)$$

where the parameters a, b are related to the low energy values of top quark mass m_t as [6]

$$\begin{aligned} a(m_t) &= 4.04704 \times 10^{-3} - 4.41909 \times 10^{-5} \left(\frac{m_t}{\text{GeV}} \right) + 1.24732 \times 10^{-7} \left(\frac{m_t}{\text{GeV}} \right)^2, \\ b(m_t) &= \exp \left[-0.979261 \left(\frac{m_t}{\text{GeV}} - 172.051 \right) \right]. \end{aligned} \quad (5)$$

The parameter c encodes the appearance of an extremum and depends on the values for top quark mass and Higgs mass. A search in the parameter space using a Monte-Carlo chain has shown [6] that even though slow-roll inflation can be realized – a result which does not hold for minimally coupled Higgs field – the resulting ratio of perturbation amplitudes is too large for any experimentally allowed values for the masses of the top quark and the Higgs boson.

Finally, by considering the energy lost to gravitational radiation by orbiting binaries and requiring the magnitude of deviations from General Relativity (GR) obtained within the NCSG context, to be less than the allowed uncertainty in the data, we imposed [7] an upper limit to the moment f_0 , which is used to specify the initial conditions on the gauge couplings. In particular, by setting $\beta^2 = (5\pi)/(48Gf_0)$ we imposed [7] the lower limit: $\beta > 7.55 \times 10^{-13} \text{ m}^{-1}$. This observational constraint may seem weak, however it is comparable to existing constraints on similar, *ad hoc*, additions to GR. Moreover, since the strongest constraint comes from systems with high orbital frequencies, future observations of rapidly orbiting binaries, relatively close to the Earth, could improve it by many orders of magnitude. Thus, by purely astrophysical observations we were able to constrain the natural length, defined through the f_0 momentum of the cut-off function f at which the noncommutative effects become dominant.

5. Conclusions

NCSG offers an elegant and purely geometric interpretation of the SM of electroweak and strong interactions. The doubling of the algebra is an essential element in order to get the gauge fields of the SM. Moreover, the NCSG classical construction carries implicit in its feature of the doubling of the algebra the seeds of quantization. The NCSG model lives at unification scale; thus it provides an excellent framework to address early universe cosmological questions, while to study astrophysical consequences one will have to go beyond the perturbative approach.

References

- [1] Chamseddine A H, Connes, A and Marcolli M 2007 Gravity and the standard model with neutrino mixing *Adv. Theor. Math. Phys.* **11** 991
- [2] Sakellariadou M 2011 Noncommutative Geometry Spectral Action as a framework for unification: Introduction and phenomenological/cosmological consequences *Int. J. Mod. Phys. D* **20** 785
- [3] Sakellariadou M, Stabile A and Vitiello G 2011 Noncommutative spectral geometry, algebra doubling and the seeds of quantization *Phys. Rev. D* **84** 045026
- [4] Nelson N and Sakellariadou M 2010 Cosmology and the Noncommutative approach to the Standard Model *Phys. Rev. D* **81** 085038
- [5] Nelson W and Sakellariadou M 2009 Natural inflation mechanism in asymptotic noncommutative geometry *Phys. Lett. B* **680** 263
- [6] Buck M, Fairbairn M and Sakellariadou M 2010 Inflation in models with Conformally Coupled Scalar fields: An application to the Noncommutative Spectral Action *Phys. Rev. D* **82** 043509
- [7] Nelson W, Ochoa J and Sakellariadou M 2010 Constraining the Noncommutative Spectral Action via Astrophysical Observations *Phys. Rev. Lett.* **105** 101602
- [8] Nelson W, Ochoa J and Sakellariadou M 2010 Gravitational Waves in the Spectral Action of Noncommutative Geometry *Phys. Rev. D* **82** 085021
- [9] Connes A 1994 *Noncommutative Geometry* (New York: Academic Press)
- [10] Connes A and Marcolli M 2009 *Noncommutative Geometry, Quantum Fields and Motives* (India: Hindustan Book Agency)
- [11] Chamseddine A H and Connes A 2007 Conceptual Explanation for the Algebra in the Noncommutative Approach to the Standard Model *Phys. Rev. Lett.* **99** 191601
- [12] Chamseddine A H and Connes 1997 The Spectral Action Principle *Commun. Math. Phys.* **186** 731
- [13] Chamseddine A H and Connes A 2006 Scale Invariance in the Spectral Action *J. Math. Phys.* **47** 063504
- [14] 't Hooft G 1999 Quantum Gravity as a Dissipative Deterministic System *Class. Quant. Grav.* **16** 3263